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Nonexistence of Best Alternating Approximations with Isolated Extrema

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Existence and nonexistence of best Chebyshev approximations by alternating families on domains X with isolated points is studied. Key factors are degeneracy, whether extrema are isolated, and whether endpoints are extrema.

Let X be a fixed compact subset of $[\alpha, \beta]$ and let $C(X)$ be the space of continuous functions on X. For $g \in C(X)$ define

$$\|g\| = \sup\{|g(x)|: x \in X\}.$$

Let F be an approximating function on $[\alpha, \beta]$ depending on a parameter varying in a set P. The Chebyshev problem on P is, given $f \in C(X)$, to find a parameter A^* minimizing $\|f - F(A, \cdot)\|$ over P. Such a parameter A^* is called best and $F(A^*, \cdot)$ is called best to f on X. We will consider only approximating functions F for which best approximation on $[\alpha, \beta]$ is characterized by alternation [7; 12, Chap. 7], which implies alternation and uniqueness on X [8] if card$(X) >$ maximum degree of F, which we henceforth assume.

We consider the case in which X has isolated points which are extrema of $f - F(A^*, \cdot)$. The case of most practical interest is where X is finite, in which case all points are isolated.

The computation of best approximations faces special difficulty in the case in which best approximations are degenerate. Special algorithms which tend to be much more elaborate and slower, e.g., the differential correction algorithm [11] or the algorithm of Belyh [2] in the rational case, must be used. The results of this paper suggest that such special algorithms may not be worthwhile since in computation we solve a perturbation of the original problem and a solution to the perturbed problem may not exist. The problem is thus computationally unstable and ill-posed.

The best known alternating approximating functions are varisolvent

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approximating functions, discussed by Rice [12, Chap. 7] (we assume that there is no problem with a nonzero constant error curve [1]).

Let (F, P) be varisolvent on $[\alpha, \beta]$. Let $F(A^*, \cdot)$, best to f on X, be of maximum degree. There exists $\varepsilon > 0$ such that $\| g - f \| < \varepsilon$ implies g has a best approximation and it is of maximum degree. This is proven by the same arguments as those of [3]. We wish to consider the case in which the best approximation is not of maximum degree.

The author's study of several alternating approximating functions on $[\alpha, \beta]$ suggests that a key property is that of *irregularity*, introduced in [5] and further exploited in [9].

DEFINITION. F is α-irregular at A if for any triple (x, y, ε), $\alpha < x < \beta$, y a real number, and $\varepsilon > 0$, there is a parameter $B \in P$ such that $|F(B, \alpha) - y| < \varepsilon$ and

$$|F(B, w) - F(A, w)| < \varepsilon, \qquad x \leqslant w \leqslant \beta.$$

Examples are given in [9]. The essential idea is that $F(B, \cdot)$ is close to $F(A, \cdot)$ except for the addition of a δ-function near α.

α an Extremum

A point is called an *extremum* of g if plus or minus the norm of g is attained there.

THEOREM 1. *Let α be isolated in X. Let F be α-irregular at A^* and α be an extremum of $f - F(A^*, \cdot) \not\equiv 0$. Then there exists a sequence $f_k \in C(X)$, $\{f_k\} \to f$, such that no best approximation exists to f_k on X.*

Proof. Assume without loss of generality that $e := f(\alpha) - F(A^*, \alpha)$ *is* >0. *Define*

$$\begin{aligned} f_k(x) &= f(x), & x &= \alpha \\ &= f(x) + [F(A^*, x) - f(x)]/k, & x &\in X \sim \alpha, \end{aligned}$$

then

$$\begin{aligned} f_k(x) - F(A^*, x) &= f(x) - F(A^*, x), & x &= \alpha \\ &= [f(x) - F(A^*, x)][1 - 1/k], & x &\in X \sim \alpha. \end{aligned}$$

By a generalization of the lemma of de la Vallée–Poussin [4], for $F(A, \cdot) \not\equiv F(A^*, \cdot)$,

$$\begin{aligned} \| f_k - F(A, \cdot) \| &> \min\{ |f_k(x_i) - F(A^*, x_i)| : i = 0, \ldots, l \} \\ &= e[1 - 1/k], \end{aligned}$$

where $\alpha = x_0$ and $\{x_0, \dots, x_l\}$ is an alternant of $f - F(A^*, \cdot)$ on X. Further by definition of irregularity, there is a sequence B_j with $F(B_j, \cdot) \to F(A^*, \cdot)$ on $X \sim \{\alpha\}$ and

$$|f_k(\alpha) - F(B_j, \alpha)| \leqslant e(1 - 1/k);$$

hence $\|f_k - F(B_j, \cdot)\| \to e(1 - 1/k)$.

Endpoints Not Extrema

Consider the case in which endpoints of $[\alpha, \beta]$ are endpoints of X. If endpoints are not extrema of $f - F(A^*, \cdot)$, best approximations may exist to all functions sufficiently near f. Consider first approximation by the family of ordinary rationals $R_1^0[\alpha, \beta]$ of ratios p/q of constants p to first degree polynomials q, $q(x) > 0$ for $\alpha \leqslant x \leqslant \beta$.

THEOREM 2. *Let $\{\alpha, \beta\} \subset X$. Let $R(A^*, \cdot)$ be best to f from $R_1^0[\alpha, \beta]$ and no endpoint be an extremum of $f - R(A^*, \cdot)$. There exists $\varepsilon > 0$, such that if $\|f - g\| < \varepsilon$, a best approximation exists to g.*

Proof. In view of previously stated results concerning maximum degree, difficulties can arise only if $R(A^*, \cdot)$ is of less than maximum degree, which implies it is zero. Select ε such that $|f(x)| + 2\varepsilon < \|f\|$ for x either endpoint. Then if $\|g - f\| < \varepsilon$, an endpoint will not be an extremum of g. The approximations zero except on one endpoint are not better than 0 to g. As adding functions zero except on one endpoint gives existence by Goldstein's theory [12, pp. 84–89], a best approximation to g exists.

Exactly the same theory holds for $F(A, x) = a_1 \exp(a_2 x)$ with [6] substituting for Goldstein's theory.

Rationals with Degeneracy $\geqslant 2$

Consider approximation by ordinary rationals $R_m^n[\alpha, \beta]$.

THEOREM 3. *Let $R(A^*, \cdot)$ be the best approximation to f on X by $R_m^n[\alpha, \beta]$ and have degeneracy two or more. If $f - R(A^*, \cdot) \not\equiv 0$ has an isolated extremum, there is $\{f_k\} \to f$ with f_k having no best approximation in $R_m^n[\alpha, \beta]$.*

Proof. In view of Theorem 1, we can assume that $f-R(A^*,\cdot)$ has an interior extremum γ. Assume without loss of generality that $e=f(\gamma)-R(A^*,\gamma)>0$. Define

$$\begin{aligned} f_k(x) &= f(x), && x=\gamma \\ &= f(x)+[R(A^*,x)-f(x)]/k, && x\neq\gamma. \end{aligned}$$

The rest of the proof is similar to that of Theorem 1 with help from the proof of Theorem 3 of [9].

Rationals Restricted to X

A problem of interest is approximation by $R_m^n(X)=\{p/q: \partial p\leqslant n, \partial q\leqslant m, q(x)>0 \text{ for } x\in X\}$. This does not fall under our earlier theory since this family may contain elements not continuous on $[\alpha,\beta]$ and best approximation is characterized by an alternation which may be nonstandard [10].

Let p/q be replaced by the equivalent p'/q' such that p' and q' are relatively prime but q' is not necessarily >0 on X, as in [10]. If q' happens to be >0 on X, the theory of [10] yields a standard alternation result and a standard de La Vallée–Poussin type result. In that case, straightforward analogues of Theorems 1 and 3 can be obtained, using the facts that p'/q' having degeneracy $\geqslant$ one implies that $p'/q'+\lambda/(x-\alpha+\mu)$ is in $R_m^n(x)$ for $\mu>0$ and p'/q' having degeneracy $\geqslant$ two implies that $p'/q'+\lambda/(x-\gamma+\mu)^2$ is in $R_m^n(X)$ for μ small. The analogue of Theorem 2 applies by the same arguments as for Theorem 2.

Part of the idea behind Theorems 1 and 3, and their analogous is to get an isolated extremum γ of $f-R(A^*,\cdot)$ and construct a sequence $\{R(A_k,\cdot)\}$ converging uniformly to $R(A^*,\cdot)$ on $X\sim\gamma$ and taking an arbitrary value on γ. This may not be possible when q' is not >0 on X. Consider the first example of [10], in which $R(A^*,x)=x/x^2=1/x$ is best in $R_2^1(X)$. Construction of the sequence $\{R(A^k,\cdot)\}$ above does not seem possible for $\gamma=\alpha$. Let $\delta=\inf\{x: x\in X, x>0\}$ be isolated and define for given λ,

$$R(A_k,x)=1/x+\lambda[1/k]/[x-\delta+1/k].$$

Both denominators above are >0 for $x\geqslant\delta$ and <0 for $x<0$, so $Q(A_k,\cdot)>0$. $R(A_k,\delta)=\lambda$ and $R(A_k,x)\to R(A^*,\cdot)$ uniformly for $x\in X\sim\delta$. We thus have a type of irregularity at δ. A similar construction is possible for $\gamma=\sup\{x: x\in X, x<0\}$. We can obtain analogues of Theorem 1 for the example with isolated extrema at γ or δ by use of the de la Vallée–Poussin type result of [10].

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